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LETTER TO THE EDITOR

Magnetic exponents of the two-dimensional q-state Potts model

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Abstract. The results of a variational renormalisation-group calculation for the magnetic exponent y_H of the two-dimensional q-state Potts model suggest a simple relationship between y_H and the exactly known critical exponent $y_T{}^{8v}$ of the eight-vertex model. The relation allows one to predict the critical and tricritical magnetic exponent δ of the q-state Potts model as a function of q.

We propose that the magnetic exponent y_H of the two-dimensional q-state Potts model is related to the thermal exponent $y_T{}^{8v}$ of the eight-vertex model along the Baxter line according to

$$(4y_H + y_T{}^{8v} - 6)(y_T{}^{8v} - 2) = -3, \tag{1}$$

when $q \leq q_c = 4$. Here $y_T{}^{8v} = (2/\pi)\cos^{-1}(\sqrt{q}/2)$, with $0 \leq y_T{}^{8v} \leq 1$ for the critical and $-1 \leq y_T{}^{8v} \leq 0$ for the tricritical phase transition of the Potts model, respectively. When $q > q_c$, the transition of the Potts model is of first order and $y_H = 2$. The exponent δ is obtained from

$$\delta = y_H/(2 - y_H). \tag{2}$$

For q equal to 4, 3, 2, and 1, the relation predicts for δ the critical values 15, 14, 15, $18\frac{1}{5}$, and the tricritical values 15, 20, $25\frac{2}{3}$, $37\frac{2}{5}$. The conjecture (1) is not based on new insight into the Potts model but rather on observations arising from numerical work, which employs the variational renormalisation-group approach to the Potts model (Nienhuis *et al* 1980), and from exactly known data for the critical δ at q equal to 2, 3, and 4. The magnetic correction exponent $y_{H,2}$ is also discussed. For $q = q_c$, the renormalisation-group calculation yields $y_{H,2} \approx \frac{7}{8}$ in agreement with a conjecture by Barber (1976).

It had been suggested that the magnetic exponent δ (or y_H) of the Potts model is independent of q (Berker *et al* 1978, den Nijs 1979, unpublished PhD Thesis). This suggestion is analogous to the conjecture by Barber and Baxter (1973) that the magnetic exponent of the eight-vertex model remains constant along the Baxter line. For $q = 2$ and 4, the value $\delta = 15$ is known exactly (for example, Barber and Baxter 1973). However, evidence that δ is not independent of q is provided by the recent exact solution of the hard-hexagon model by Baxter (1980). For this model, believed to be in the universality class of the $q = 3$ Potts model (Alexander 1975), Baxter obtained $\delta = 14$. Earlier evidence is the result of $\delta \approx 18$ for the $q = 1$ Potts model (Dasgupta 1976, Gaunt and Sykes 1976). Our calculation confirms the variation of the magnetic exponent with q and provides a simple formula describing the variation.

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Nienhuis *et al* (1979, 1980) proposed a novel renormalisation-group transformation for the Potts model that revealed the topology of the renormalisation-group flow diagram. For $q < q_c$, lines of critical and tricritical fixed points exist, that merge at q_c. Along the fixed lines the Potts exponents vary continuously as functions of q. A calculation using the variational renormalisation-group method of Kadanoff (1975) confirmed this picture for the thermal exponent y_T and yielded very satisfactory agreement with exact results for the critical exponents and the critical value $q_c = 4$ (Nienhuis *et al* 1980). The same approximation method with the same weight function is employed here to determine the magnetic exponents of the Potts model. The model is studied embedded in the larger space of Potts–lattice-gas Hamiltonians

$$-\beta\mathcal{H} = \sum_{\langle i,j\rangle} t_i t_j (K + J\delta_{s_i,s_j}) - \Delta \sum_i t_i \tag{3}$$

with the symmetry breaking term

$$-\beta\mathcal{H}' = H \sum_i t_i \delta_{1,s_i} + \sum_{\langle i,j\rangle} t_i t_j [L(\delta_{1,s_i} + \delta_{1,s_j}) + M\delta_{1,s_i}\delta_{1,s_j}]. \tag{4}$$

The lattice gas variable t_i equals unity if a Potts spin $s_i = 1, 2, \ldots, q$ occupies lattice site i and is zero otherwise. The chemical potential Δ governs the concentration of vacancies.

The results of the calculation are shown in three figures, each exhibiting three sets of data. The renormalisation-group equations have two fixed-line solutions. One exhibits the full topology of critical and tricritical branches (broken curve) and has a free energy that assumes a maximum as a function of the variational parameters. At $q = 2$, the result reduces to that of Burkhardt (1976). For reasons not fully understood no results were found for small q on the critical branch. The second solution yields critical exponents (dotted curve) in good agreement with exact results at small q but fails to yield the first-order transition at large q. This fixed line lies in the pure Potts model subspace but has a free energy that assumes a saddle point. The solution is identical with Dasgupta's (1976, 1977), who did not consider the possibility of vacancies.

Figure 1 exhibits a plot of the magnetic exponent y_H versus the thermal exponent y_T of the Potts model along the critical ($y_T < 1\cdot5$) and tricritical ($y_T > 1\cdot5$) fixed lines. These

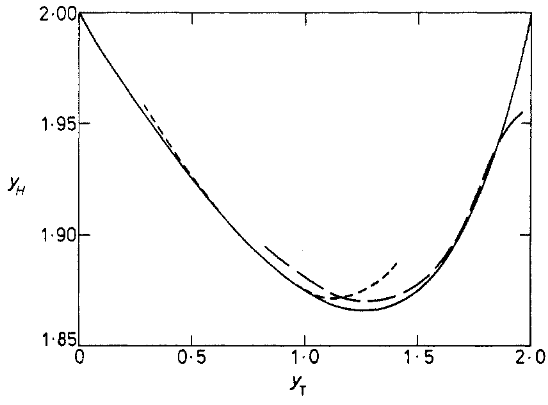

Figure 1. Magnetic exponent y_H versus thermal exponent y_T for the q-state Potts model from variational renormalisation-group calculation (broken and dotted curves) and conjecture equation (5).

numerical results (broken and dotted curves) suggested to us that y_H and y_T satisfy the simple relation (full curve)

$$(4y_H + y_T - 7)(y_T - 3) = -3. \tag{5}$$

Combining this result with the den Nijs conjecture $(y_T - 3)(y_T^{8v} - 2) = 3$ (den Nijs 1979, Nienhuis *et al* 1979), one obtains the result of equation (1). This new conjecture reproduces correctly all exactly known critical values of δ. Furthermore, it assumes a particularly simple form when used to compute the exponent $\beta = (2 - y_H)/y_T$ for the Potts model,

$$\beta = \frac{(1 + y_T^{8v})}{12}, \tag{6}$$

as communicated to us by Pearson (1980, private communication). The variation of y_H with q is shown in figure 2. The full, broken and dotted curves have the same meaning as in figure 1.

The variation of the second magnetic exponent $y_{H,2}$ with q is shown in figure 3. At q_c, it assumes to within 0·2% the value $\frac{7}{8}$ conjectured by Barber (1976) for the

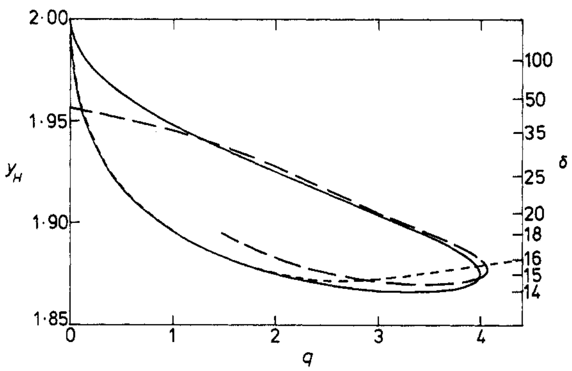

Figure 2. Critical and tricritical magnetic exponents y_H (lower and upper branch, respectively) of the Potts model from variational renormalisation-group calculation (broken and dotted curves) and conjecture equation (1).

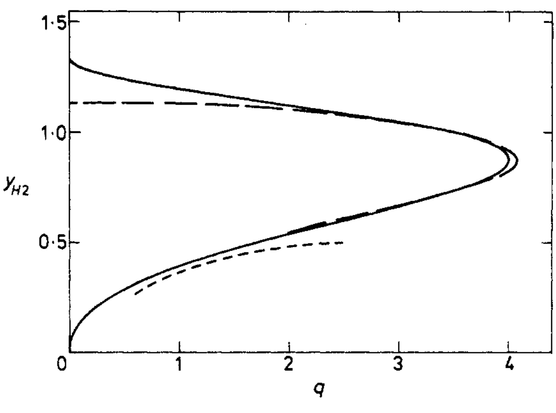

Figure 3. Second magnetic exponent $y_{H,2}$ for the critical and tricritical Potts transition (lower and upper branch, respectively) from variational renormalisation-group calculation (broken and dotted curves) and equation (7).

corresponding exponent of the Baxter–Wu model. The relation

$$(4y_{H,2}+y_T{}^{8v}-6)=5, \qquad (7)$$

fits the numerical data rather well and yields $y_{H,2}(q=0)=0$ (Lubensky 1978). No other exact results for $y_{H,2}$ at $q<q_c$ are known. Relations of the form $y_i=y_i(y_T{}^{8v})$ should exist for any exponent of the Potts model as they depend only on q.

In summary, our variational renormalisation-group approach provides accurate numerical results for the critical and tricritical exponents of the Potts model. It provides the basis for our conjecture on the variation of y_H with q and leads us to believe that the thermal and magnetic exponents of the Potts model are now known exactly.

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